

Advanced Microeconomics

Ordinal preference theory

Harald Wiese

University of Leipzig

Part A. Basic decision and preference theory

1. Decisions in strategic (static) form
2. Decisions in extensive (dynamic) form
3. **Ordinal preference theory**
4. Decisions under risk

Ordinal preference theory

Overview

1. The vector space of goods and its topology
2. Preference relations
3. Axioms: convexity, monotonicity, and continuity
4. Utility functions
5. Quasi-concave utility functions and convex preferences
6. Marginal rate of substitution

The vector space of goods

Assumptions

- ▶ households are the only decision makers;
- ▶ finite number ℓ of goods defined by
 - ▶ place
 - ▶ time
 - ▶ contingencies
- ▶ example: an apple of a certain weight and class to be delivered
 - ▶ in Leipzig
 - ▶ on April 1st, 2015,
 - ▶ if it does not rain the day before

The vector space of goods

Exercise: Linear combination of vectors

Problem

Consider the vectors $x = (x_1, x_2) = (2, 4)$ and $y = (y_1, y_2) = (8, 12)$. Find $x + y$, $2x$ and $\frac{1}{4}x + \frac{3}{4}y$!

The vector space of goods

Notation

- ▶ For $\ell \in \mathbb{N}$:

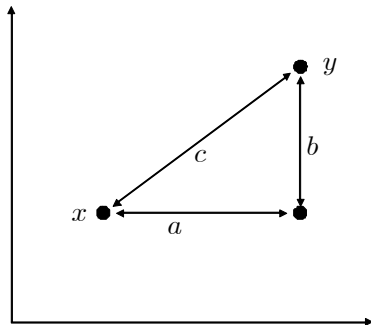
$$\mathbb{R}^\ell := \{(x_1, \dots, x_\ell) : x_g \in \mathbb{R}, g = 1, \dots, \ell\}.$$

- ▶ $0 \in \mathbb{R}^\ell$ – null vector $(0, 0, \dots, 0)$;
 - ▶ vectors are called points (in $\mathbb{R}^\ell$).
- ▶ Positive amounts of goods only:

$$\mathbb{R}_+^\ell := \{x \in \mathbb{R}^\ell : x \geq 0\}$$

- ▶ Vector comparisons:
 - ▶ $x \geq y :\Leftrightarrow x_g \geq y_g$ for all g from $\{1, 2, \dots, \ell\}$;
 - ▶ $x > y :\Leftrightarrow x \geq y$ and $x \neq y$;
 - ▶ $x \gg y :\Leftrightarrow x_g > y_g$ for all g from $\{1, 2, \dots, \ell\}$.

Distance between x and y



euclidian
distance: c

city-block
distance: $a+b$

infinity
distance: $\max(a, b)$

Distance

Definition

In $\mathbb{R}^\ell$:

- ▶ euclidian (or 2-) norm:

$$\|x - y\| := \|x - y\|_2 := \sqrt{\sum_{g=1}^{\ell} (x_g - y_g)^2}$$

- ▶ infinity norm: $\|x - y\|_\infty := \max_{g=1, \dots, \ell} |x_g - y_g|$

- ▶ city-block norm: $\|x - y\|_1 := \sum_{g=1}^{\ell} |x_g - y_g|$

Distance

Exercise

Problem

What is the distance (in $\mathbb{R}^2$) between $(2, 5)$ and $(7, 1)$, measured by the 2-norm $\|\cdot\|_2$ and by the infinity norm $\|\cdot\|_\infty$?

Distance and balls

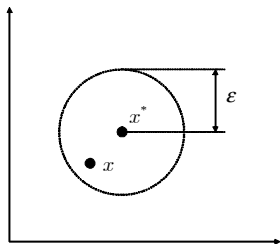
Ball

Definition

Let $x^* \in \mathbb{R}^\ell$ and $\varepsilon > 0$. $\Rightarrow$

$$K = \{x \in \mathbb{R}^\ell : \|x - x^*\| < \varepsilon\}$$

– (open) ε -ball with center x^* .



- ▶ $\|x - x^*\| = \varepsilon$ holds for all x on the circular line;
- ▶ K – all the points within

Problem

Assuming the goods space $\mathbb{R}_+^2$, sketch three 1-balls with centers $(2, 2)$, $(0, 0)$ and $(2, 0)$, respectively.

Distance and balls

Boundedness

Definition

A set M is bounded if there exists an ε -ball K such that $M \subseteq K$.

Example

The set $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$ is not bounded.

Sequences and convergence

Sequence

Definition

A sequence $(x^j)_{j \in \mathbb{N}}$ in $\mathbb{R}^\ell$ is a function $\mathbb{N} \rightarrow \mathbb{R}^\ell$.

Examples

- ▶ In $\mathbb{R}$: $1, 2, 3, 4, \dots$ or $x^j := j$;
- ▶ In $\mathbb{R}^2$:
 - ▶ $(1, 2), (2, 3), (3, 4), \dots$
 - ▶ $\left(1, \frac{1}{2}\right), \left(1, \frac{1}{3}\right), \left(1, \frac{1}{4}\right), \left(1, \frac{1}{5}\right), \dots$

Sequences and convergence

Convergence

Definition

$(x^j)_{j \in \mathbb{N}}$ in $\mathbb{R}^\ell$ converges towards $x \in \mathbb{R}^\ell$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\|x^j - x\| < \varepsilon \text{ for all } j > N$$

holds.

- ▶ Convergent – a sequence that converges towards some $x \in \mathbb{R}^\ell$.

Examples

- ▶ $1, 2, 3, 4, \dots$ is not convergent towards any $x \in \mathbb{R}$;
- ▶ $1, 1, 1, 1, \dots$ converges towards 1;
- ▶ $1, \frac{1}{2}, \frac{1}{3}, \dots$ converges towards zero.

Sequences and convergence

Lemma 1

Lemma

Let $(x^j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{R}^\ell$.

- ▶ If $(x^j)_{j \in \mathbb{N}}$ converges towards x and $y \Rightarrow x = y$.
- ▶ $(x^j)_{j \in \mathbb{N}} = (x_1^j, \dots, x_\ell^j)_{j \in \mathbb{N}}$ converges towards $(x_1, \dots, x_\ell)$ iff x_g^j converges towards x_g for every $g = 1, \dots, \ell$.

Problem

Convergent?

$$(1, 2), (1, 3), (1, 4), \dots$$

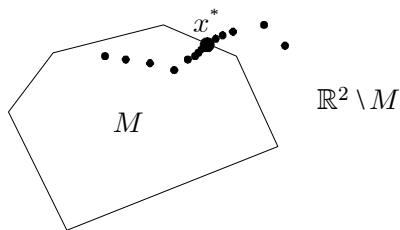
or

$$\left(1, \frac{1}{2}\right), \left(1, \frac{1}{3}\right), \left(1, \frac{1}{4}\right), \left(1, \frac{1}{5}\right), \dots$$

Boundary point

Definition

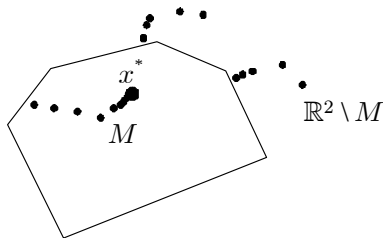
A point $x^* \in \mathbb{R}^\ell$ is a boundary point of $M \subseteq \mathbb{R}^\ell$ iff there is a sequence of points in M and another sequence of points in $\mathbb{R}^\ell \setminus M$ so that both converge towards x^* .



Interior point

Definition

A point in M that is not a boundary point is called an interior point of M .



Note: Instead of $\mathbb{R}^\ell$, we can consider alternative sets, for example $\mathbb{R}_+^\ell$.

Closed set

Definition

A set $M \subseteq \mathbb{R}^\ell$ is closed iff every converging sequence in M with convergence point $x \in \mathbb{R}^\ell$ fulfills $x \in M$.

Problem

Are the sets

- ▶ $\{0\} \cup (1, 2)$,
- ▶ $K = \{x \in \mathbb{R}^\ell : \|x - x^*\| < \varepsilon\}$,
- ▶ $K = \{x \in \mathbb{R}^\ell : \|x - x^*\| \leq \varepsilon\}$

closed?

Preference relations

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Relations and equivalence classes

Definition

Definition

Relations R (write xRy) might be

- ▶ complete (xRy or yRx for all x, y)
- ▶ transitive (xRy and yRz implies xRz for all x, y, z)
- ▶ reflexive (xRx for all x);

Problem

For any two inhabitants from Leipzig, we ask whether one is the father of the other. Fill in "yes" or "no":

property is the father of
reflexive
transitive
complete

Preference relations and indifference curves

Preference relation

Definition

- ▶ (weak) preference relation $\succsim$ – a relation on $\mathbb{R}_+^\ell$ that is
 - ▶ complete ($x \succsim y$ or $y \succsim x$ for all x, y)
 - ▶ transitive ($x \succsim y$ and $y \succsim z$ implies $x \succsim z$ for all x, y, z) and
 - ▶ reflexive ($x \succsim x$ for all x);
- ▶ indifference relation:

$$x \sim y :\Leftrightarrow x \succsim y \text{ and } y \succsim x;$$

- ▶ strict preference relation:

$$x \prec y :\Leftrightarrow x \succsim y \text{ and not } y \succsim x.$$

Preference relations and indifference curves

Exercise

Problem

Fill in:

<i>property</i>	<i>indifference</i>	<i>strict preference</i>
<i>reflexive</i>		
<i>transitive</i>		
<i>complete</i>		

Preference relations and indifference curves

transitivity of strict preference

We want to show

$$x \prec y \wedge y \prec z \Rightarrow x \prec z$$

Proof: $x \prec y$ implies $x \succsim y$, $y \prec z$ implies $y \succsim z$.

Therefore, $x \prec y \wedge y \prec z$ implies $x \succsim z$.

Assume $z \succsim x$. Together with $x \prec y$, transitivity implies $z \succsim y$, contradicting $y \prec z$. Therefore, we do not have $z \succsim x$, but $x \prec z$.

Preference relations and indifference curves

Better and indifference set

Definition

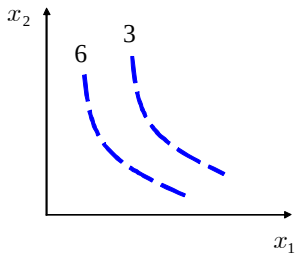
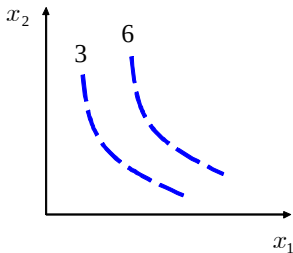
Let $\succsim$ be a preference relation on $\mathbb{R}_+^\ell$. $\Rightarrow$

- ▶ $B_y := \{x \in \mathbb{R}_+^\ell : x \succ y\}$ – better set B_y of y ;
- ▶ $W_y := \{x \in \mathbb{R}_+^\ell : x \precsim y\}$ – worse set W_y of y ;
- ▶ $I_y := B_y \cap W_y = \{x \in \mathbb{R}_+^\ell : x \sim y\}$ – y 's indifference set;
- ▶ indifference curve – the geometric locus of an indifference set.

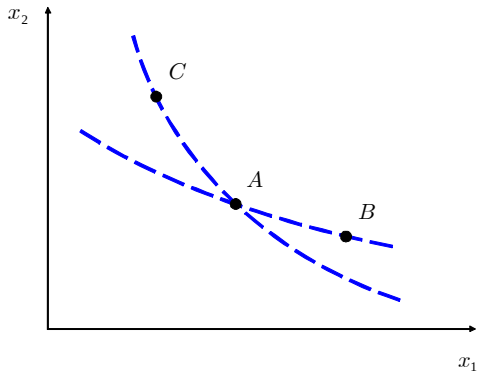
Preference relations and indifference curves

Indifference curve

numbers to indicate preferences:



Indifference curves must not intersect



Two different
indifference curves,
Thus $C \succ B$

But
 $C \sim A \wedge A \sim B \Rightarrow C \sim B$

Contradiction!

Preference relations and indifference curves

Exercise

Problem

Sketch indifference curves for a goods space with just 2 goods and, alternatively,

- ▶ *good 2 is a bad,*
- ▶ *good 1 represents red matches and good 2 blue matches,*
- ▶ *good 1 stands for right shoes and good 2 for left shoes.*

Preference relations and indifference curves

Lexicographic preferences

In two-good case:

$$x \succsim_{lex} y :\Leftrightarrow x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$

Problem

What do the indifference curves for lexicographic preferences look like?

Axioms: convexity, monotonicity, and continuity

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Convex preferences

Convex combination

Definition

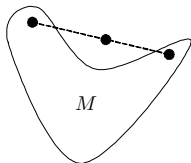
Let x and y be elements of $\mathbb{R}^\ell$. $\Rightarrow$

$$kx + (1 - k)y, k \in [0, 1]$$

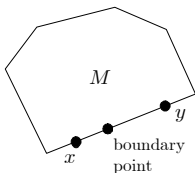
– the convex combination of x and y .

Convex preferences

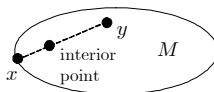
Convex and strictly convex sets



not convex



convex,
but not strictly convex



strictly convex

Definition

A set $M \subseteq \mathbb{R}^\ell$ is convex if for any two points x and y from M , their convex combination is also contained in M .

A set M is strictly convex if for any two points x and y from M , $x \neq y$,

$$kx + (1 - k)y, k \in (0, 1)$$

is an interior point of M for any $k \in (0, 1)$.

Convex preferences

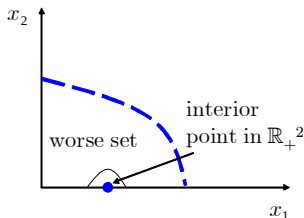
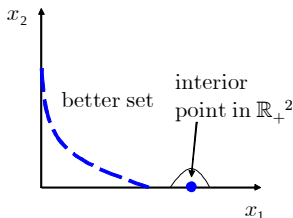
Convex and concave preference relation

Definition

A preference relation $\succsim$ is

- ▶ convex if all its better sets B_y are convex,
- ▶ strictly convex if all its better sets B_y are strictly convex,
- ▶ concave if all its worse sets W_y are convex,
- ▶ strictly concave if all its worse sets W_y are strictly convex.

Preferences are defined on $\mathbb{R}_+^\ell$ (!):

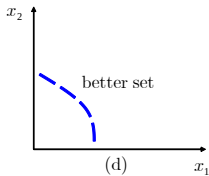
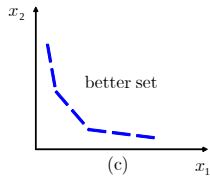
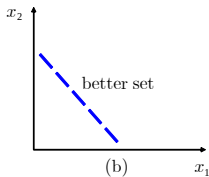
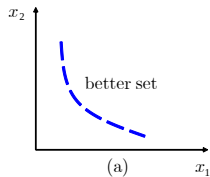


Convex preferences

Exercise

Problem

Are these preferences convex or strictly convex?



Monotonicity of preferences

Monotonicity

Definition

A preference relation $\succsim$ obeys:

- ▶ weak monotonicity if $x > y$ implies $x \succsim y$;
- ▶ strict monotonicity if $x > y$ implies $x \succ y$;
- ▶ local non-satiation at y if in every ε -ball with center y a bundle x exists with $x \succ y$.

Problem

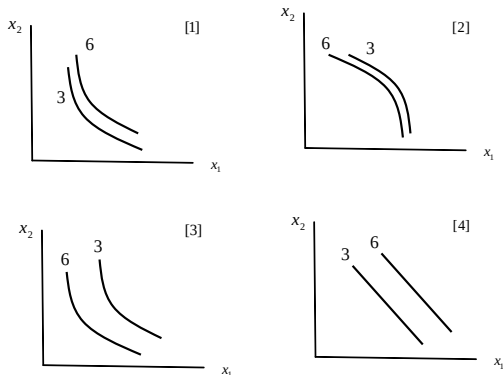
Sketch the better set of $y = (y_1, y_2)$ in case of weak monotonicity!

Exercise: Monotonicity and convexity

Which of the properties

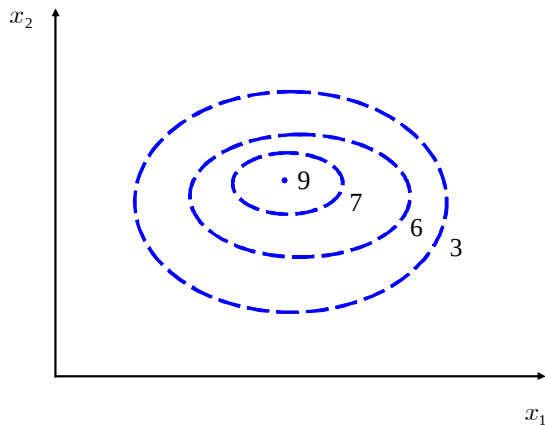
- ▶ (strict) monotonicity and/or
- ▶ (strict) convexity

do the preferences depicted by the indifference curves in the graphs below satisfy?



Monotonicity of preferences

Bliss point



Continuous preferences

Definition

A preference relation $\succsim$ is continuous if for all $y \in \mathbb{R}_+^\ell$ the sets

$$W_y = \left\{ x \in \mathbb{R}_+^\ell : x \succsim y \right\}$$

and

$$B_y = \left\{ x \in \mathbb{R}_+^\ell : y \succsim x \right\}$$

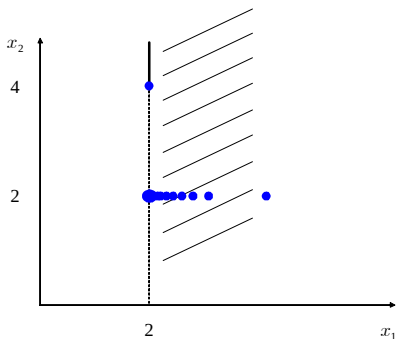
are closed.

Lexicographic preferences are not continuous

Consider the sequence

$$(x^j)_{j \in \mathbb{N}} = \left(2 + \frac{1}{j}, 2\right) \rightarrow (2, 2)$$

All its members belong to the better set of $(2, 4)$.
But $(2, 2)$ does not.



Utility functions

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Utility functions

Definition

Definition

For an agent $i \in N$ with preference relation $\succsim^i$,

$$U^i : \mathbb{R}_+^\ell \mapsto \mathbb{R}$$

– utility function if

$$U^i(x) \geq U^i(y) \Leftrightarrow x \succsim^i y, x, y \in \mathbb{R}_+^\ell$$

holds.

- ▶ U^i represents the preferences $\succsim^i$;
- ▶ ordinal utility theory.

Utility functions

Examples of utility functions

Examples

- ▶ Cobb-Douglas utility functions (weakly monotonic):

$$U(x_1, x_2) = x_1^a x_2^{1-a} \text{ with } 0 < a < 1;$$

- ▶ perfect substitutes:

$$U(x_1, x_2) = ax_1 + bx_2 \text{ with } a > 0 \text{ and } b > 0;$$

- ▶ perfect complements:

$$U(x_1, x_2) = \min(ax_1, bx_2) \text{ with } a > 0 \text{ and } b > 0.$$

Utility functions

Dixit-Stiglitz preferences for love of variety

$$U(x_1, \dots, x_\ell) = \left(\sum_{j=1}^{\ell} x_j^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \text{ with } \varepsilon > 1$$

where $\bar{X} := \sum_{j=1}^{\ell} x_j$ implies

$$\begin{aligned} U\left(\frac{\bar{X}}{\ell}, \dots, \frac{\bar{X}}{\ell}\right) &= \left(\sum_{j=1}^{\ell} \left(\frac{\bar{X}}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\sum_{j=1}^{\ell} \bar{X}^{\frac{\varepsilon-1}{\varepsilon}} \left(\frac{1}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left(\ell \bar{X}^{\frac{\varepsilon-1}{\varepsilon}} \left(\frac{1}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} = \ell^{\frac{\varepsilon}{\varepsilon-1}} \bar{X}^{\frac{1}{\ell}} = \ell^{\frac{\varepsilon}{\varepsilon-1}-1} \bar{X} = \ell^{\frac{1}{\varepsilon-1}} \bar{X} \end{aligned}$$

and hence, by $\frac{\partial U}{\partial \ell} > 0$, a love of variety.

Utility functions

Exercises

Problem

Draw the indifference curve for perfect substitutes with $a = 1$, $b = 4$ and the utility level 5!

Problem

Draw the indifference curve for perfect complements with $a = 1$, $b = 4$ (a car with four wheels and one engine) and the utility level for 5 cars! Does x_1 denote the number of wheels or the number of engines?

Equivalent utility functions

Definition (equivalent utility functions)

Two utility functions U and V are called equivalent if they represent the same preferences.

Lemma (equivalent utility functions)

Two utility functions U and V are equivalent iff there is a strictly increasing function $\tau : \mathbb{R} \rightarrow \mathbb{R}$ such that $V = \tau \circ U$.

Problem

Which of the following utility functions represent the same preferences?

a) $U_1(x_1, x_2, x_3) = (x_1 + 1)(x_2 + 1)(x_3 + 1)$

b) $U_2(x_1, x_2, x_3) = \ln(x_1 + 1) + \ln(x_2 + 1) + \ln(x_3 + 1)$

c) $U_3(x_1, x_2, x_3) = -(x_1 + 1)(x_2 + 1)(x_3 + 1)$

d) $U_4(x_1, x_2, x_3) = -[(x_1 + 1)(x_2 + 1)(x_3 + 1)]^{-1}$

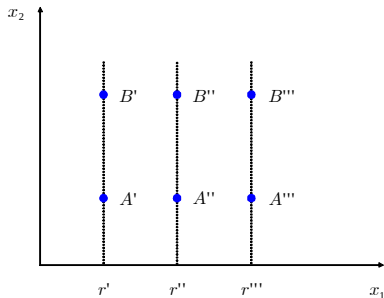
e) $U_5(x_1, x_2, x_3) = x_1 x_2 x_3$

Existence

Existence is not guaranteed

Assume a utility function U for lexicographic preferences:

- ▶ $U(A') < U(B') < U(A'') < U(B'') < U(A''') < U(B''')$;
- ▶ within $(U(A'), U(B'))$ at least one rational number (q') etc.
- ▶ $q' < q'' < q'''$;
- ▶ injective function $f : [r', r'''] \rightarrow Q$;
- ▶ not enough rational numbers;
- ▶ contradiction $\longrightarrow$ no utility function for lexicographic preferences



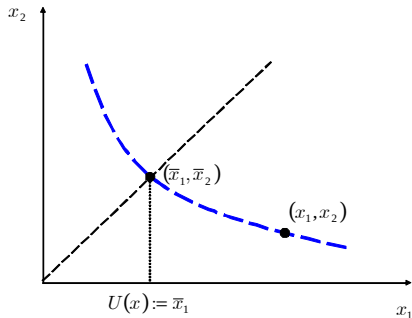
Existence

Existence of a utility function for continuous preferences

Theorem

If the preference relation $\succsim^i$ of an agent i is continuous, there is a continuous utility function U^i that represents $\succsim^i$.

Wait a second for the definition of a continuous utility function, please!



Existence

Exercise

Problem

Assume a utility function U that represents the preference relation $\succsim$.

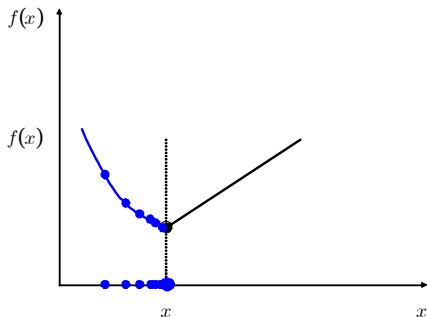
Can you express weak monotonicity, strict monotonicity and local non-satiation of $\succsim$ through U rather than $\succsim$?

Continuous functions

Definition

Definition

$f : \mathbb{R}^\ell \rightarrow \mathbb{R}$ is
continuous at $x \in \mathbb{R}^\ell$ iff
for every $(x^j)_{j \in \mathbb{N}}$ in $\mathbb{R}^\ell$
that converges towards
 x , $(f(x^j))_{j \in \mathbb{N}}$
converges towards $f(x)$.



Fix $x \in \mathbb{R}^\ell$.

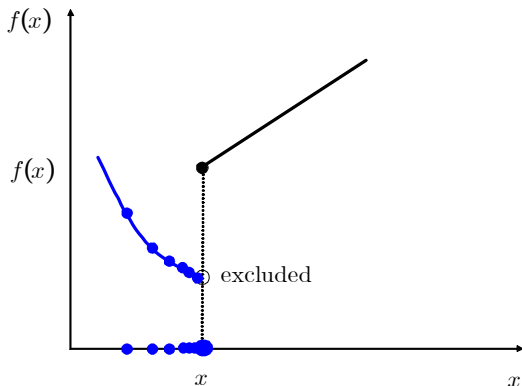
Consider sequences in the domain $(x^j)_{j \in \mathbb{N}}$ that converge towards x .

Convergence of sequences in the range $(f(x^j))_{j \in \mathbb{N}}$ towards $f(x)$
 $\longrightarrow f$ continuous at x

No convergence or convergence towards $y \neq f(x) \longrightarrow$ no

Continuous functions

Counterexample



Specific sequence in the domain $(x^j)_{j \in \mathbb{N}}$ that converges towards x but sequence in the range $(f(x^j))_{j \in \mathbb{N}}$ does not converge towards $f(x)$

Quasi-concave utility functions and convex preferences

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Quasi-concavity

Definition

- ▶ $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$ is quasi-concave if

$$f(kx + (1 - k)y) \geq \min(f(x), f(y))$$

holds for all $x, y \in \mathbb{R}^\ell$ and all $k \in [0, 1]$.

- ▶ f is strictly quasi-concave if

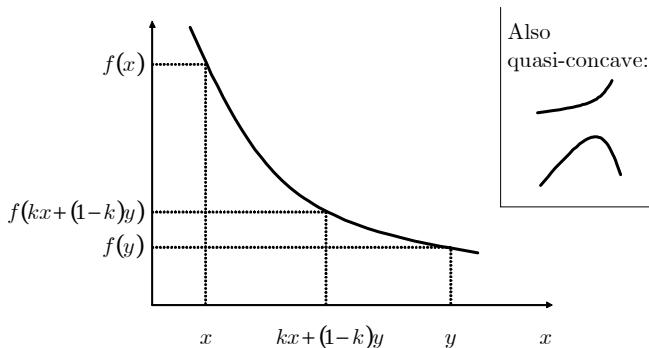
$$f(kx + (1 - k)y) > \min(f(x), f(y))$$

holds for all $x, y \in \mathbb{R}^\ell$ with $x \neq y$ and all $k \in (0, 1)$.

Note: quasi-concave functions need not be concave (to be introduced later).

Quasi-concavity

Examples

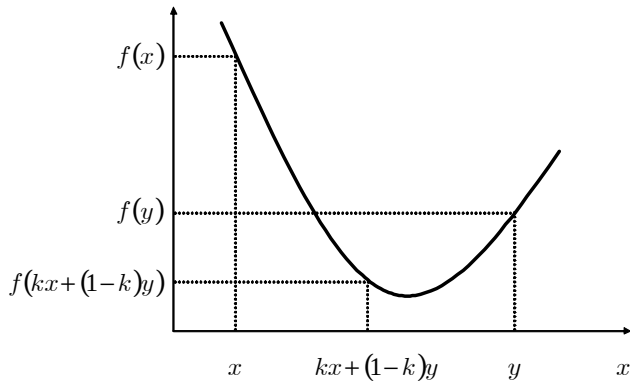


Example

Every monotonically increasing or decreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ is quasi-concave.

Quasi-concavity

A counter example



Better and worse sets, indifference sets

Definition

Let U be a utility function on $\mathbb{R}_+^\ell$.

- ▶ $B_{U(y)} := B_y = \{x \in \mathbb{R}_+^\ell : U(x) \geq U(y)\}$ – the better set B_y of y ;
- ▶ $W_{U(y)} := W_y = \{x \in \mathbb{R}_+^\ell : U(x) \leq U(y)\}$ – the worse set W_y of y ;
- ▶ $I_{U(y)} := I_y = B_y \cap W_y = \{x \in \mathbb{R}_+^\ell : U(x) = U(y)\}$ – y 's indifference set (indifference curve) I_y .

Concave indifference curve

Definition

Let U be a utility function on $\mathbb{R}_+^\ell$.

- ▶ I_y is concave if $U(x) = U(y)$ implies

$$U(kx + (1 - k)y) \geq U(x)$$

for all $x, y \in \mathbb{R}_+^\ell$ and all $k \in [0, 1]$.

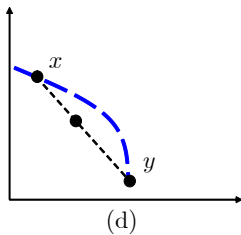
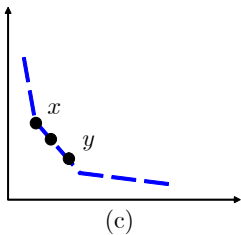
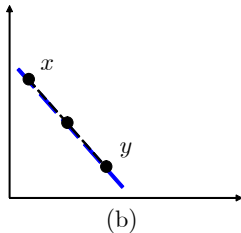
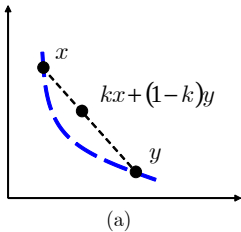
- ▶ I_y is strictly concave if $U(x) = U(y)$ implies

$$U(kx + (1 - k)y) > U(x)$$

for all $x, y \in \mathbb{R}_+^\ell$ with $x \neq y$ and all $k \in (0, 1)$.

Concave indifference curve

Examples and counter examples



Lemma

Lemma

Let U be a continuous utility function on $\mathbb{R}_+^\ell$. A preference relation $\succsim$ is convex iff:

- ▶ all the indifference curves are concave, or
- ▶ U is quasi-concave.

U 's better sets strictly convex $\iff U$ strictly quasi-concave $\implies U$ quasi-concave $\iff U$'s better sets convex



U 's better sets strictly convex and local nonsatiation $\implies U$'s indifference curves strictly concave $\implies U$'s indifference curves concave

Marginal rate of substitution

Overview

1. The vector space of goods and its topology
2. Preference relations
3. Axioms: convexity, monotonicity, and continuity
4. Utility functions
5. Quasi-concave utility functions and convex preferences
6. **Marginal rate of substitution**

Marginal rate of substitution

Mathematics: Differentiable functions

Definition

Let $f : M \rightarrow \mathbb{R}$ be a real-valued function with open domain $M \subseteq \mathbb{R}^\ell$.

f is differentiable if all the partial derivatives

$$f_i(x) := \frac{\partial f}{\partial x_i} \quad (i = 1, \dots, \ell)$$

exist and are continuous.

$$f'(x) := \begin{pmatrix} f_1(x) \\ f_2(x) \\ \dots \\ f_\ell(x) \end{pmatrix}$$

– f' 's derivative at x .

Marginal rate of substitution

Mathematics: Adding rule

Theorem

Let $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$ be a differentiable function and let $g_1, \dots, g_\ell$ be differentiable functions $\mathbb{R} \rightarrow \mathbb{R}$. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

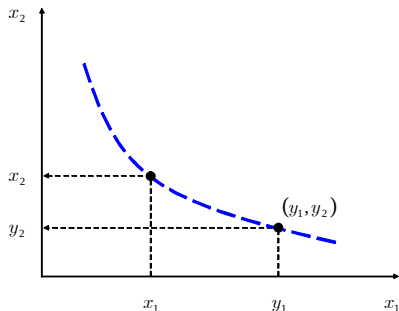
$$F(x) = f(g_1(x), \dots, g_\ell(x)).$$

$\Rightarrow$

$$\frac{dF}{dx} = \sum_{i=1}^{\ell} \frac{\partial f}{\partial g_i} \frac{dg_i}{dx}.$$

Marginal rate of substitution

Economics



- ▶ $I_y = \{(x_1, x_2) \in \mathbb{R}_+^2 : (x_1, x_2) \sim (y_1, y_2)\}.$
- ▶ $I_y : x_1 \mapsto x_2.$

Marginal rate of substitution

Definition and exercises

Definition

If the function I_y is differentiable and if preferences are monotonic,

$$\left| \frac{dI_y(x_1)}{dx_1} \right|$$

– the MRS between good 1 and good 2 (or of good 2 for good 1).

Problem

What happens if good 2 is a bad?

Marginal rate of substitution

Perfect substitutes

Problem

Calculate the MRS for perfect substitutes ($U(x_1, x_2) = ax_1 + bx_2$ with $a > 0$ and $b > 0$.)!

- ▶ Solve $ax_1 + bx_2 = k$ for x_2 !
- ▶ Form the derivative of x_2 with respect to x_1 !
- ▶ Take the absolute value!

Marginal rate of substitution

Lemma 1

Lemma

Let $\succsim$ be a preference relation on $\mathbb{R}_+^\ell$ and let U be the corresponding utility function.

If U is differentiable, the MRS is defined by:

$$MRS(x_1) = \left| \frac{dl_y(x_1)}{dx_1} \right| = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}.$$

$\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}$ – marginal utility.

Marginal rate of substitution

Lemma: proof

Proof.

- ▶ $U(x_1, l_y(x_1))$ – constant along indifference curve;
- ▶ differentiating $U(x_1, l_y(x_1))$ with respect to x_1 (adding rule):

$$0 = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} \frac{dl_y(x_1)}{dx_1}.$$

- ▶ $\left| \frac{dl_y(x_1)}{dx_1} \right|$ can be found even if l_y were not given explicitly (implicit-function theorem).



Problem

Again: What is the MRS in case of perfect substitutes?

Marginal rate of substitution

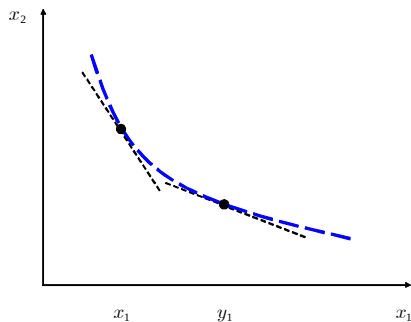
Lemma 2

Lemma

Let U be a differentiable utility function and I_y an indifference curve of U .

I_y is concave iff the MRS is a decreasing function in x_1 .

- ▶ $x_1 < y_1$;
- ▶ $MRS(x_1) > MRS(y_1)$.



Marginal rate of substitution

Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^a x_2^{1-a}, 0 < a < 1$$

$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a x_1^{a-1} x_2^{1-a}}{(1-a) x_1^a x_2^{-a}} = \frac{a}{1-a} \frac{x_2}{x_1}.$$

- ▶ Cobb-Douglas preferences are monotonic so that an increase of x_1 is associated with a decrease of x_2 along an indifference curve.
- ▶ Therefore, Cobb-Douglas preferences are convex (Cobb-Douglas utility functions are quasi-concave).

Further exercises

Problem 1

Define strict anti-monotonicity. Sketch indifference curves for each of the four cases:

	strict monotonicity	strict anti-monotonicity
strict convexity		
strict concavity		

Problem 2

(Strictly) monotonic, (strictly) convex or continuous?

- (a) $U(x_1, x_2) = x_1 \cdot x_2$,
- (b) $U(x_1, x_2) = \min \{a \cdot x_1, b \cdot x_2\}$ where $a, b > 0$ holds,
- (c) $U(x_1, x_2) = a \cdot x_1 + b \cdot x_2$ where $a, b > 0$ holds,
- (d) lexicographic preferences

Further exercises

Problem 3 (difficult)

Let U be a continuous utility function representing the preference relation $\succsim$ on $\mathbb{R}_+^\ell$. Show that $\succsim$ is continuous as well. Also, give an example for a continuous preference relation that is represented by a discontinuous utility function. Hint: Define a function U' that differs from U for $x = 0$, only